



TITLE:

Lower Bounds of Growth Order of Solutions of Schrodinger Equations with Homogeneous Potentials (SPECTRAL AND SCATTERING THEORY AND RELATED TOPICS)

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Lower bounds of Growth order of solutions of
Schrödinger equations with homogeneous potentials

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We shall study on the asymptotic behavior as $|x|$ tends to ∞ of the solution $u(x) \in H_{loc}^2(\Omega)$ satisfying the following equation.

$$(1) \quad -\Delta u(x) + q(x)u(x) = \lambda u(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^n (n \geq 3),$$

where λ is a positive constant and $\Omega \supset E_{R_0} = \{x \in \mathbb{R}^n; |x| > R_0\}$.

The conditions to be imposed on the potential $q(x)$ are following ones.

Assumption 1. The potential $q(x)$ is real valued and satisfies the following inequality:

$$(2) \quad |x| < Dq, \tilde{x} > \leq (-2\gamma)q(x) \quad \text{for } x \in \Omega,$$

where γ is some constant, Dq is a gradient of $q(x)$, $\tilde{x} = \frac{x}{|x|}$, and $\langle \cdot, \cdot \rangle$ denotes the inner product of \mathbb{C}^n .

Assumption 2. For $q(x)$, we have

$$(3) \quad \sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq 1} \frac{|q(y)|^2}{|x-y|^{n-4+\mu}} dy < +\infty$$

for some constant $\mu > 0$.

Assumption 3. The unique continuation property holds.

Remark. If $q(x)$ is a homogeneous functions of x of degree -2γ , then (2) is satisfied. So we cannot expect that $q(x)$ decays uniformly at infinity.

Example. The potential of the Schrödinger operator of an atom (or ion) consisting of a nucleus with charge $+Z$ and m electrons given by

$$(4) \quad q(x) = - \sum_{k=1}^m \frac{Z}{r_k} + \sum_{1 \leq k < l \leq m} \frac{1}{r_{kl}},$$

where

$$r_k^2 = \sum_{l=0}^2 |x_{3k-l}|^2, \quad r_{kj}^2 = \sum_{l=0}^2 |x_{3k-l} - x_{3j-l}|^2, \text{ is a}$$

homogeneous function of $x \in \mathbb{R}^{3m}$ of degree -1 (i.e. $\gamma = \frac{1}{2}$)

We can admit $u(x)$ be a real valued function. Our aim is to take α in the following estimation (5) for the not identically vanishing solution $u(x)$ of (1) as large as possible.

$$(5) \quad \liminf_{r \rightarrow \infty} r^{-\alpha} \int_{S_r} (\text{some form of } u) dS > 0,$$

where $S_r = \{ x \in \mathbb{R}^n; |x| = r \}$.

To this end we introduce $v(x)$ by $v(x) = |x|^{\frac{n-1}{2}} u(x)$, and then $v(x)$ satisfies the following equation

$$(6) \quad -\Delta v + \frac{n-1}{|x|} \langle Dv, \tilde{x} \rangle + (q(x) + \tilde{q}(x) - \lambda) v(x) = 0 \text{ in } \Omega,$$

where

$$(7) \quad \tilde{q}(x) = \frac{(n-1)(n-3)}{4|x|^2}.$$

According to the calculation used in Ikebe-Uchiyama [4], we have by (6) and integration by parts

$$(8) \quad 0 = \int_{B_{Sr}} \left\{ \text{the left side of (6)} \right\} \left\{ 2|x|^\alpha \langle Dv, \tilde{x} \rangle - \right. \\ \left. - k|x|^{\alpha-1} v \right\} dx$$

$$= \left(\int_{S_r} - \int_{S_s} \right) \{ \text{some form of } v \} dS + \int_{B_{sr}} \{ \text{another form of } v \} dx,$$

where $B_{sr} = \{ x \in \mathbb{R}^n; s < |x| < r \} \subset \Omega$. Here parameter k is introduced according to the method used in Agmon [1]. We define the surface integral on S_r in (8) as $F(r, \alpha; k)$. Namely

$$(9) \quad F(r, \alpha; k) = \int_{S_r} \left\{ 2 \langle Dv, \tilde{x} \rangle^2 - |Dv|^2 + (\lambda - q(x) - \tilde{q}(x)) |v|^2 - \right. \\ \left. - \frac{k}{|x|} \langle Dv, \tilde{x} \rangle v + \frac{(\alpha + n - 2)}{2} \frac{k}{|x|^2} |v|^2 \right\} |x|^\alpha dS.$$

Then let us rewrite (8), and we have

$$(10) \quad F(r, \alpha; k) - F(s, \alpha; k) = \int_{B_{sr}} \left[(3 - \alpha - n - k) (|Dv|^2 - \langle Dv, \tilde{x} \rangle^2) + \right. \\ \left. + (\alpha + n - k - 1) \langle Dv, \tilde{x} \rangle^2 + \{ (\alpha + n + k - 1) \lambda - (\alpha + n + k - 1) q(x) - \right. \\ \left. - |x| \langle Dq, \tilde{x} \rangle - (\alpha + n + k - 3) \tilde{q}(x) + \right. \\ \left. + \frac{k}{2} - (\alpha + n - 2)(\alpha + n - 3) \frac{1}{|x|^2} \} |v|^2 \right] |x|^{\alpha-1} dx, \text{ for } r > s > 0,$$

where $B_{sr} \subset \Omega$.

By (10) we have

Lemma 1. If $0 \leq \gamma \leq 1$, we have for any $r > s > 0$ satisfying $B_{sr} \subset \Omega$

$$F(r, \alpha_0; \gamma) \geq F(s, \alpha_0; \gamma),$$

where $\alpha_0 = 1 - n + \gamma$.

Noting the following relation

$$(11) \quad F(r, \alpha_0; \gamma) = r^\gamma \int_{S_r} \left[2 \langle Du, \tilde{x} \rangle^2 - |Du|^2 + \right. \\ \left. + \frac{(n-1-\gamma)}{|x|} \langle Du, \tilde{x} \rangle u + (\lambda - q(x)) |u|^2 + \right.$$

$$+ \frac{(1-\gamma)(n-1-\gamma)}{2|x|^2} |u|^2 \Big] ds$$

and $u(x) \in H^2_{loc}(\Omega)$, we have

Lemma 2. If $\Omega = \mathbb{R}^n$ and $0 < \gamma \leq 1$, we have

$$\lim_{s \rightarrow 0} F(s, \alpha_0; \gamma) = 0.$$

Noting that $u(x)$ is a not identically vanishing solution of (1), we have by (10) and Lemma 2

Lemma 3. If $\Omega = \mathbb{R}^n$ and $0 < \gamma \leq 1$, there exists some $r_0 > 0$ such that

$$F(r_0, \alpha_0; \gamma) > 0.$$

So by Lemma 1 and Lemma 3, we have

Theorem 1. If $\Omega = \mathbb{R}^n$, and if u is a not identically vanishing solution of (1), then we have

$$\liminf_{R \rightarrow \infty} R^{\gamma-1} \int_{|x| \leq R} |u(x)|^2 dx > 0, \text{ when } 0 < \gamma < 1,$$

and

$$\liminf_{R \rightarrow \infty} (\log R)^{-1} \int_{|x| \leq R} |u(x)|^2 dx > 0, \text{ when } \gamma = 1.$$

Hereafter we only assume $\Omega \supset E_{R_0}$. In order to prove Lemma 3, we introduce the function $w_m(x)$ by

$$(12) \quad w_m(x) = e^{mf(|x|)} v(x),$$

and the form $F(r, \alpha, \beta; m; k; f)$ by

$$(13) \quad F(r, \alpha, \beta, m; k; f(r)) = \int_{S_r} [2 \langle Dw_m, \tilde{x} \rangle^2 - |Dw_m|^2 - \\ - \frac{k}{|x|} \langle Dw_m, \tilde{x} \rangle w_m + \left\{ \lambda - q(x) + \tilde{q}(x) + m^2 f'^2 - mf'' - \right.$$

$$\begin{aligned}
& - |x|^{\beta-\alpha} + \frac{k}{2} (\alpha+n-2) \frac{1}{|x|^2} + \frac{km}{|x|} f' \} |w_m|^2 \Big] |x|^\alpha dS \\
& = e^{2mf(r)} \left[F(r, \alpha; k) + \int_{S_r} \{ 2mf' \langle Dv, \tilde{x} \rangle v + \right. \\
& \quad \left. + (2m^2 f'^2 - mf'' - |x|^{\beta-\alpha}) |v|^2 \} |x|^\alpha dS \right].
\end{aligned}$$

In order to show $F(r, \alpha_0; \gamma) > 0$, it is sufficient to prove

$$\begin{aligned}
& F(r, \alpha_0, \beta, m; \gamma; f(r)) > 0 \text{ and } \int_{S_r} \{ 2mf' \langle Dv, \tilde{x} \rangle v + \\
& + (2m^2 f'^2 - mf'' - |x|^{\beta-\alpha}) |v|^2 \} |x|^\alpha dS < 0 \text{ for some } \beta, m \text{ and } \\
& f(r).
\end{aligned}$$

To investigate the property of $F(r, \alpha, \beta, m; k; f(r))$, we apply the calculation, which is similar to (8) or (10), to $F(r, \alpha, \beta, m; k; f(r))$, noting the identity

$$\begin{aligned}
(14) \quad & 2 \int_{B_{sr}} \langle Dw, \tilde{x} \rangle w |x|^\beta dx = \int_{S_r} |w|^2 |x|^\beta dS - \int_{S_s} |w|^2 |x|^\beta dS - \\
& - (n+\beta-1) \int_{B_{sr}} |w|^2 |x|^{\beta-1} dx.
\end{aligned}$$

So we have

$$\begin{aligned}
(15) \quad & F(r, \alpha, \beta, m; k; f) - F(s, \alpha, \beta, m; k; f) = \\
& = \int_{B_{sr}} \left[(3-\alpha-n-k) (|Dw_m|^2 - \langle Dw_m, \tilde{x} \rangle^2) + \right. \\
& + (\alpha+n-k-1+4m|x|f') \langle Dw_m, \tilde{x} \rangle^2 - 2|x|^{\beta-\alpha+1} \langle Dw_m, \tilde{x} \rangle w_m + \\
& + \{ (\alpha+n+k-1)\lambda - (\alpha+n+k-1)q(x) - |x| \langle Dq, \tilde{x} \rangle - \\
& - (\alpha+n+k-3)\tilde{q}(x) + (\alpha+n+k-1)m^2 f'^2 + 2m^2 |x| f' f'' - \\
& - (\alpha+n-1)mf'' + k(\alpha+n-2) \frac{1}{|x|} mf' - |x| mf''' - \\
& \left. - (\beta+n-1)|x|^{\beta-\alpha} + \frac{k}{2}(\alpha+n-2)(\alpha+n-3) \frac{1}{|x|^2} \} |w_m|^2 \right] |x|^{\alpha-1} dx.
\end{aligned}$$

Here we put $\alpha = \alpha_0$, $\beta = \alpha_0 - \delta$, $k = \gamma$ and $f(r) = r^\varepsilon$. Then we have

Lemma 4. If $\varepsilon \geq 2(1 - \delta)$, $\varepsilon > 0$, $\varepsilon > 1 - \gamma$ and $\delta > 0$, then there exist some $R_1 > R_0$ and some $m_0 > 0$ such that for any $r > s \geq R_1$ and for any $m \geq m_0$ we have

$$F(r, \alpha_0, \alpha_0 - \delta, m; \gamma; r^\varepsilon) \geq F(s, \alpha_0, \alpha_0 - \delta, m; \gamma; s^\varepsilon).$$

Unique continuation property leads that there exists some sequence $\{r_1\}_{1=1,2,\dots}$ such that $R_1 < r_1 < r_2 < \dots < r_1 < r_{1+1} < \dots$, $\lim_{l \rightarrow \infty} r_1 = \infty$ and $\int_{S_{r_1}} |v|^2 ds > 0$ ($l=1,2,\dots$).

Since in the right hand term of (13) the coefficient of e^{2mr^ε} is a quadratic form of m , in which the coefficient of m^2 is positive, we have

Lemma 5. There exists some constant $m_1 \geq m_0$ such that

$$F(r_1, \alpha_0, \alpha_0 - \delta, m_1; \gamma; r_1^\varepsilon) > 0.$$

By Lemma 4 and Lemma 5, we have

Lemma 6. Under the same conditions imposed ⁱⁿ Lemma 4, we have

$$F(r, \alpha_0, \alpha_0 - \delta, m_1; \gamma; r^\varepsilon) > 0 \text{ for any } r \geq r_1.$$

Now the problem we consider is reduced to the following Case I or Case II:

Case I. There exists some sequence $\{r'_1\}_{1=1,2,\dots}$ such that

$$r_1 < r'_1 < r'_2 < \dots < r'_1 < r'_{1+1} < \dots, \lim_{l \rightarrow \infty} r'_1 = \infty$$

$$\text{and } \int_{S_{r'_1}} \langle Dv, \tilde{x} \rangle v ds \leq 0 \quad (l=1,2,\dots),$$

Case II. There exists some $R_2 \geq r_1$ such that $\int_{S_r} \langle Dv, \tilde{x} \rangle v ds > 0$

$$\text{(i.e. } \frac{d}{dr} \int_{S_r} |u|^2 ds > 0) \quad \text{for any } r \geq R_2.$$

By (13), we have

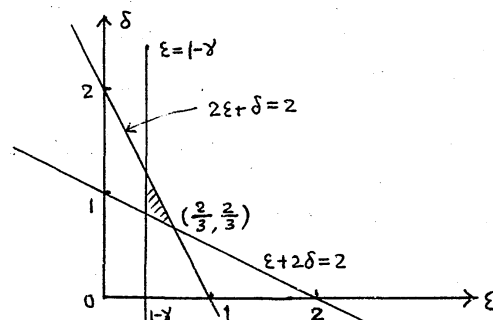
Lemma 7. We assume the same conditions imposed in Lemma 4. Moreover if Case I holds and if $2\varepsilon - 2 < -\delta$, there exists some $R_3 \geq R_1$ such that $F(R_3, \alpha_0; \gamma) > 0$.

Lemma 8. If $\frac{1}{3} < \gamma \leq 1$, and if Case I holds, then the statement of Lemma 7 holds.

If Case II holds, $\int_{S_r} |u|^2 ds$ is

a monotone increasing function.

Consequently under either Case I or Case II, we have



Theorem 2. If $\Omega \supset E_{R_0}$, and if u is a not identically vanishing solution of (1), then we have

$$\liminf_{R \rightarrow \infty} R^{\gamma-1} \int_{R_0 \leq |x| \leq R} |u(x)|^2 dx > 0, \quad \text{when } \frac{1}{3} < \gamma < 1,$$

and

$$\liminf_{R \rightarrow \infty} (\log R)^{-1} \int_{R_0 \leq |x| \leq R} |u(x)|^2 dx > 0, \quad \text{when } \gamma = 1.$$

Corollary. If $\Omega = R^n$ and $0 < \gamma \leq 1$, or if $\Omega \supset E_{R_0}$ and

$\frac{1}{3} < \gamma \leq 1$, the Schrödinger operator appearing on the left side of (1) has no positive eigenvalues.

Remark. Weidmann [5] has (not explicitly) shown under the conditions $\Omega = \mathbb{R}^n$ and $0 < \gamma \leq \frac{1}{2}$,

$$\liminf_{R \rightarrow \infty} R^{2\gamma-1} \int_{|x| \leq R} |u(x)|^2 dx > 0, \quad (0 < \gamma < \frac{1}{2})$$

and

$$\liminf_{R \rightarrow \infty} (\log R)^{-1} \int_{|x| \leq R} |u(x)|^2 dx > 0, \quad (\gamma = \frac{1}{2}).$$

It has shown that the Schrödinger operator appearing in (1) has no positive eigenvalue by Weidmann [5] under the conditions

$\Omega = \mathbb{R}^n$ and $0 < \gamma \leq \frac{1}{2}$, Weidmann [6] under the conditions

$\Omega = \mathbb{R}^n$ and $\frac{1}{2} \leq \gamma < 1$ and Agmon [2], [3] under $\Omega \supset E_{R_0}$

and $\frac{1}{2} \leq \gamma < 1$.

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